

Metric Convergence from Parabolic Higgs Bundle Moduli Spaces to Hyperpolygon Spaces

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Hyperkähler Manifolds

Definition

A *hyperkähler* manifold is a $4n$ -dimensional manifold with a triple of kähler structures $(X, g, I, J, K, \omega_I, \omega_J, \omega_K)$ such that I, J, K satisfy the quaternionic relations.

Equivalently, (X, g) has holonomy contained in $\mathrm{Sp}(n) = \mathrm{Sp}(n, \mathbb{C}) \cap \mathrm{SU}(2n)$. In particular, X is Ricci-flat.

The hyperkähler condition is very rigid. The compact hyperkähler 4-manifolds are:

- Compact tori $\mathbb{C}^2/\Gamma$ for a lattice $\Gamma \cong \mathbb{Z}^4$
- K3 surfaces (simply connected)

Recipe for Constructing Hyperkähler Manifolds

Any complex vector space has a natural kähler structure (V, I, g, ω) arising from the dot product and multiplication by i .

The cotangent bundle $X = T^*V$ of a complex vector space additionally has a holomorphic/algebraic symplectic form

$$\Omega((u_1, w_1), (u_2, w_2)) = w_1 u_2 - w_2 u_1.$$

Along with the standard kähler structure on X , we get a hyperkähler structure by decomposing $\Omega = \omega_J + i\omega_K$.

Recipe for Constructing Hyperkähler Manifolds

Suppose $G \subset G_{\mathbb{C}}$ act on a hyperkähler space (X, I, ω, Ω) , respectively preserving ω and Ω .

Suppose further the action admits moment maps $\mu_{\mathbb{R}} : X \rightarrow \mathfrak{g}^*$ and $\mu_{\mathbb{C}} : X \rightarrow \mathfrak{g}_{\mathbb{C}}^*$. Given elements $\beta \in Z(\mathfrak{g}^*)$, $\tau \in Z(\mathfrak{g}_{\mathbb{C}}^*)$, one constructs the *hyperkähler quotient* $X \mathrel{////}_{(\beta, \tau)} G$ in two ways:

Moduli of solutions to $\mu_{\mathbb{C}} = \tau$ and $\mu_{\mathbb{R}} = \beta$ up to G -equivalence:

$$X \mathrel{////}_{(\beta, \tau)} G = (\mu_{\mathbb{C}}^{-1}(\tau) \cap \mu_{\mathbb{R}}^{-1}(\beta)) / G$$

- $\mu_{\mathbb{R}} = \beta$ is often difficult to solve
- Gives information about full geometric structure $(X \mathrel{////} G, I, g, \omega, \Omega)$

Moduli of β -stable solutions to $\mu_{\mathbb{C}} = \tau$ up to $G_{\mathbb{C}}$ -equivalence:

$$X \mathrel{////}_{(\beta, \tau)} G = (\mu^{-1}(\tau))^{\beta\text{-st}} / G_{\mathbb{C}}$$

- $\mu_{\mathbb{C}} = \tau$ and β -stability are purely algebraic conditions, easy to classify
- Gives information about holomorphic symplectic structure $(X \mathrel{////} G, I, \Omega)$

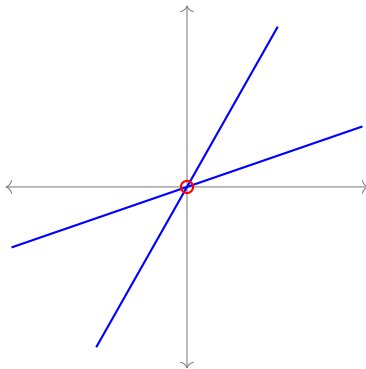
Kempf-Ness-type theorems state that

$$p \in \mu_{\mathbb{C}}^{-1}(\tau) \text{ is } \beta\text{-stable} \iff \exists! g \in G_{\mathbb{C}} / G : g \cdot p \in \mu_{\mathbb{R}}^{-1}(\beta).$$

$\mathbb{P}^1$ Example

Consider $\mathbb{C}^\times \curvearrowright \mathbb{C}^2$. The orbit $\{0\}$ is unstable.

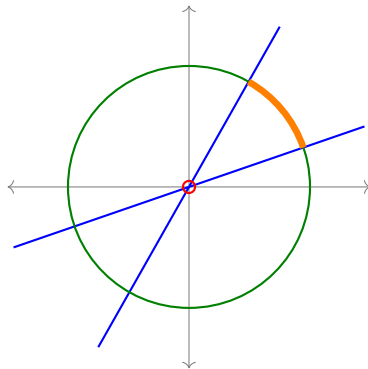
$U(1) \curvearrowright \mathbb{C}^2$ admits the moment map $\mu_{\mathbb{R}}(x) = \frac{i}{2}|x|^2$



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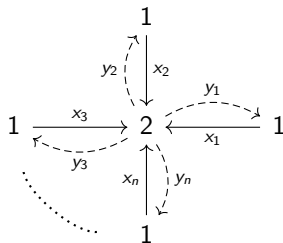
$U(1) \curvearrowright \mathbb{C}^2$ admits the moment map $\mu_{\mathbb{R}}(x) = \frac{i}{2}|x|^2 = \beta$



Restriction to the β -level set gives a well-defined notion of distance by taking paths orthogonal to the $U(1)$ -orbits (suppressed in the 2D image).

Hyperpolygon Space

Let Q be the quiver shown to the right. The space $X = \text{Rep } Q$ is isomorphic to $T^* \text{Hom}(\mathbb{C}^n, \mathbb{C}^2) \cong T^* \mathbb{C}^{2n}$ and has a standard hyperkähler structure $(X, g, I, \omega_I, \Omega_I)$.



The groups $G = \text{SU}(2) \times \text{U}(1)^n$ and $G_{\mathbb{C}} = \text{SL}(2, \mathbb{C}) \times \text{GL}(1, \mathbb{C})^n$ act on X and respectively preserve ω_I and Ω_I . The actions are hamiltonian, and admit moment maps $\mu_{\mathbb{R}} : X \rightarrow \mathfrak{g}^*$ and $\mu_{\mathbb{C}} : X \rightarrow \mathfrak{g}_{\mathbb{C}}^*$.

The moment maps decompose

$$\mu_{\mathbb{C}} = (\mu_{\text{SL}(2, \mathbb{C})}, \mu_{\text{GL}(1, \mathbb{C})^n}) : X \rightarrow \mathfrak{g}_{\mathbb{C}}^* = \mathfrak{sl}(2, \mathbb{C})^* \times (\mathbb{C}^n)^*$$

$$\mu_{\mathbb{R}} = (\mu_{\text{SU}(2)}, \mu_{\text{U}(1)^n}) : X \rightarrow \mathfrak{g}^* = \mathfrak{su}(2)^* \times (\mathfrak{u}(1)^n)^*$$

Hyperpolygon Space

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$$\mu_{\mathbb{R}} = (\mu_{\mathrm{SU}(2)}, \mu_{U(1)^n}) : X \rightarrow \mathfrak{g}^* = \mathfrak{su}(2)^* \times (\mathfrak{u}(1)^n)^*$$

$$\mu_{\mathrm{SL}(2,\mathbb{C})}(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^n (x_i y_i)_0, \quad \mu_{\mathrm{GL}(1,\mathbb{C})^n}(\mathbf{x}, \mathbf{y}) = (y_1 x_1, \dots, y_n x_n)$$

$$\mu_{\mathrm{SU}(2)}(\mathbf{x}, \mathbf{y}) = \frac{1}{2} \sum_i (x_i x_i^\dagger)_0 - (y_i^\dagger y_i)_0, \quad \mu_{U(1)^n} = \frac{1}{2} (|x_i|^2 - |y_i|^2)_{i=1}^n.$$

(A_0 denotes the trace-free part $A - \frac{1}{2} \mathrm{tr}(A) \mathrm{Id}$).

Hyperpolygon Space

Take $\tau = 0 \in Z(\mathfrak{g}_{\mathbb{C}}^*)$ and $\vec{\beta} \in Z(\mathfrak{g}^*) \cong \mathbb{R}^n$. The hyperkähler quotient is

$$\mathcal{X}(\vec{\beta}) = X \underset{(0, \vec{\beta})}{////} G = \left(\mu_{\mathbb{C}}^{-1}(0) \cap \mu_{\mathbb{R}}^{-1}(\vec{\beta}) \right) / G.$$

Thus $\mathcal{X}(\vec{\beta})$ is the *moduli space of solutions to $\mu_{\mathbb{C}} = 0$ and $\mu_{\mathbb{R}} = \vec{\beta}$ up to G -equivalence*.

Via the Kempf-Ness theorem, we can also regard $\mathcal{X}(\vec{\beta})$ as the *moduli space of $\vec{\beta}$ -stable representations up to $G_{\mathbb{C}}$ -equivalence*.

Hyperpolygon Space

The moment map equations have a geometric interpretation as “telescoping polygons” in $\mathfrak{su}(2) \cong \mathbb{R}^3$: Set $v_i := (x_i x_i^\dagger)_0$ and $w_i := (y_i^\dagger y_i)_0$.

$$\mu_{\mathrm{SL}(2,\mathbb{C})} = \sum_{i=1}^n (x_i y_i)_0 = 0$$

$$\mu_{\mathrm{GL}(1,\mathbb{C})^n, i} = y_i x_i = 0$$

$$\mu_{\mathrm{SU}(2)} = \frac{1}{2} \sum_i (x_i x_i^\dagger)_0 - (y_i^\dagger y_i)_0 = 0$$

$$\mu_{U(1)^n, i} = \frac{1}{2} (|x_i|^2 - |y_i|^2) = \beta_i$$



v_i and w_i are opposite,

$$(|v_i| - |w_i|) = \sqrt{2}\beta_i$$

$$\sum v_i + w_i = 0$$

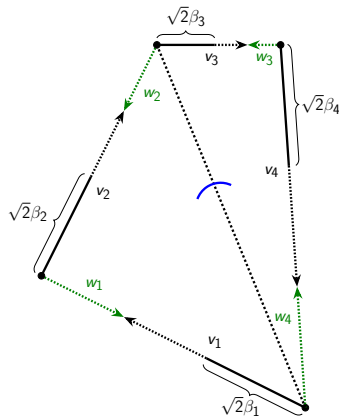


Figure: A hyperpolygon displayed as vectors in $\mathfrak{su}(2) \cong \mathbb{R}^3$

Hyperpolygon Space

Hyperpolygon space has real dimension $4(n - 3)$ [Kon02] and can be fibered over $\mathbb{C}^{n-3}$. Pictured below is the $n = 4$ hyperpolygon space:

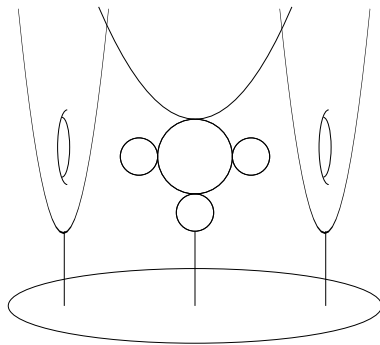


Figure: $n = 4$ Hyperpolygon Space

Parabolic Higgs Bundles

Let $C = \mathbb{CP}^1$ and $D = \{p_1, \dots, p_n\}$, and let $E = C \times \mathbb{C}^2$ be the trivial bundle with a *parabolic structure* $(\mathcal{F}, \vec{\alpha})$:

$$\begin{array}{ccccc} 0 & \subset & F_i & \subset & E_{p_i} \\ 1 & > & 1 - \alpha_i & > & \alpha_i \end{array} \quad (i = 1, \dots, n)$$

We call $\vec{\alpha} \in (0, \frac{1}{2})^n$ the *parabolic weights*.

A hermitian metric h on E is *adapted* to $(\mathcal{F}, \vec{\alpha})$ if the growth rate filtration on the stalks of E at p_i matches the filtration induced by $(\mathcal{F}_i, \alpha_i)$.

$$\text{Ex: } F_i = \langle e_1 \rangle, \quad h \sim \begin{pmatrix} |z - p_i|^{2(1-\alpha_i)} & |z - p_i|^{2\alpha_i} \\ |z - p_i|^{2\alpha_i} & |z - p_i|^{2\alpha_i} \end{pmatrix}$$

Parabolic Higgs Bundles

Let $\mathcal{E} = (E, \bar{\partial}_{\mathcal{E}}, \mathcal{F}, \vec{\alpha})$ be a parabolic bundle with a holomorphic structure.

A *strongly parabolic Higgs field* on $\mathcal{E}$ is a section $\varphi \in H^0(C, \text{End}(E) \otimes K_{\mathbb{CP}^1}(D))$ whose residues $\varphi_i \in \mathfrak{sl}(E_{p_i})$ are nilpotent with respect to $\mathcal{F}_i$:

$$\varphi = \sum_{i=1}^n \frac{\varphi_i}{z - p_i} dz \quad \sum_{i=1}^n \varphi_i = 0.$$

The parabolic weights determine a notion of $\vec{\alpha}$ -stability for a Higgs bundle.

We define $\mathcal{M}^{\text{Higgs}}(\vec{\alpha})$ to be the moduli space of $\vec{\alpha}$ -stable Higgs bundles up to gauge equivalence. This gives $\mathcal{M}^{\text{Higgs}}(\vec{\alpha})$ the structure of a holomorphic symplectic manifold.

Parabolic Higgs Bundles

Theorem (Non-Abelian Hodge Correspondence (NAHC))

Fix $R > 0$. A parabolic $SL(2, \mathbb{C})$ -Higgs bundle $(E, \bar{\partial}_{\mathcal{E}}, \mathcal{F}, \vec{\alpha}, \varphi)$ is stable if and only if there exists a unique adapted metric h_R (up to unitary equivalence) solving Hitchin's equations

$$\begin{aligned}\mu_{\mathbb{C}}(\mathcal{E}, \varphi) &= \bar{\partial}_{\mathcal{E}} \varphi = 0 \\ \mu_{\mathbb{R}}(\mathcal{E}, \varphi) &= R^{-1} F_{(\bar{\partial}_{\mathcal{E}}, h_R)}^{\perp} + R[\varphi^{\dagger h_R}, \varphi] = 0\end{aligned}$$

Such an h_R is called the *harmonic metric* for $(E, \bar{\partial}_{\mathcal{E}}, \mathcal{F}, \vec{\alpha}, \varphi)$.

The hyperkähler reduction process gives the moduli space an $\mathbb{R}_{>0}$ -family of hyperkähler structures

$$\mathcal{M}_R(\vec{\alpha}) = (\mathcal{M}^{\text{Higgs}}(\vec{\alpha}), \Omega, g_R).$$

The Embedding

Godinho and Mandini [GM11] produce an embedding $T_R : \mathcal{X}(\vec{\beta}) \hookrightarrow \mathcal{M}_R(\vec{\alpha})$ for $\vec{\beta}, \vec{\alpha}$ in appropriate chambers: given $(\mathbf{x}, \mathbf{y}) \in \mathcal{X}(\vec{\beta})$, the flag is $F_i = \langle x_i \rangle$ and the Higgs field residues are $\varphi_i = x_i y_i$.

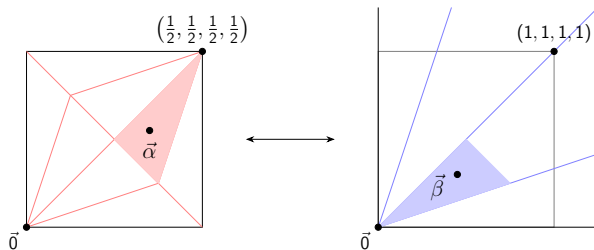


Figure: Chamber Structures

This embedding is open and dense, and respects the holomorphic symplectic structures but not the full hyperkähler metrics.

Metric Degeneration

For $n = 4$, the *Hitchin fibration* is a map $\mathcal{H} : \mathcal{M}_R(\vec{\alpha}) \rightarrow \mathbb{C}$ given by $(\mathcal{E}, \varphi) \mapsto \det \varphi$.

The fibers are generically holomorphic Lagrangian tori of volume $4\pi^2/R$, but the singular fiber $\mathcal{H}^{-1}(0)$ is five 2-spheres which generate $H_2(\mathcal{M}_R(\vec{\alpha}), \mathbb{Z})$.

Taking $R \rightarrow 0$ enlarges the fibers while shrinking the base.

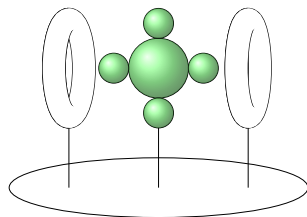


Figure: Hitchin Fibration
when $n = 4$

Metric Degeneration

On the other hand, varying $\vec{\alpha}$ within its chamber also modifies the hyperkähler structure without changing the notion of $\vec{\alpha}$ -stability.

This is reflected in the period data, also called the *Torelli numbers*, which are calculated by integrating $\omega_{I,R}$ on generators of $H_2(\mathcal{M}_R(\vec{\alpha}), \mathbb{C})$.

The chamber walls are the values of $\vec{\alpha} \in (0, \frac{1}{2})^n$ for which the respective Torelli numbers vanish, resulting in singularities in $\mathcal{M}_R(\vec{\alpha})$.

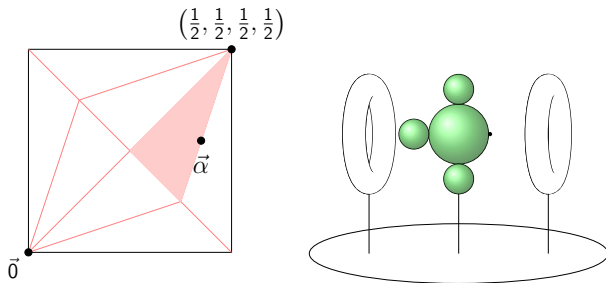


Figure: Chamber Structures

Metric Degeneration

By letting $\alpha_i(R) = \frac{1}{2} - R\beta_i$, we produce a family of moduli spaces $\mathcal{M}_R := \mathcal{M}_R(\vec{\alpha}(R))$ in which four of the five Torelli numbers are fixed, while the last grows without bound as $R \rightarrow 0$.

Theorem (Fredrickson-Y; Heller-Heller-Meneses)

As $R \rightarrow 0$, the family of metrics g_R converges pointwise to the metric $2\pi g_{\text{HP}}$ on the restriction to the image of $\mathcal{T}_R : \mathcal{X}(\vec{\beta}) \hookrightarrow \mathcal{M}_R(\vec{\alpha}(R))$. In other words, $\lim_{R \rightarrow 0} \mathcal{T}_R^*(g_R) = 2\pi g_{\text{HP}}$.

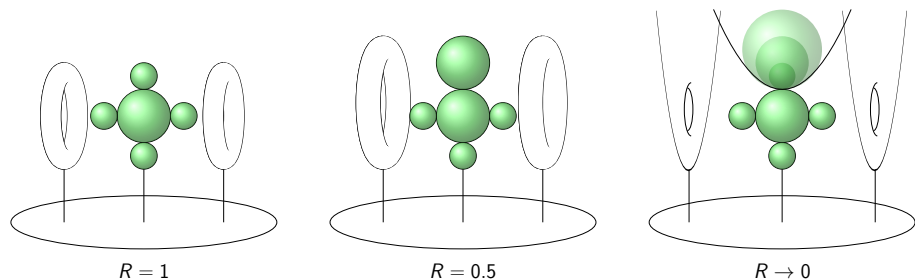


Figure: Degenerate Limit as $R \rightarrow 0$, ($n = 4$ case)

Gravitational Instantons

Definition

A *gravitational instanton* is a complete noncompact hyperkähler 4-manifold satisfying the finite energy condition

$$\int |\mathrm{Rm}|^2 < \infty.$$

G. Chen and X. Chen [CC21; CC17; CC20] provide a full classification according to the asymptotic behavior at the noncompact ends.

Gravitational Instantons

At infinity, a gravitational instanton X has the form $(\mathbb{R}^{4-m} \times \mathbb{T}^m) / \Gamma$ for $m = 0, 1, 2, 3$ and $\Gamma \subset \mathrm{SU}(2)$. Depending on m , we say X is:

ALE $m = 0$

ALF $m = 1$

ALG^(*) $m = 2$

ALH^(*) $m = 3$

(the star appears if $|\mathrm{Rm}| \notin O(r^{-2+\varepsilon})$ for any $\varepsilon > 0$.)

When $n = 4$, hyperpolygon space $\mathcal{X}(\vec{\beta})$ is an ALE- D_4 gravitational instanton [Nak94], and the Higgs bundle moduli space $\mathcal{M}_R(\vec{\alpha})$ is an ALG- D_4 instanton [FMSW21].

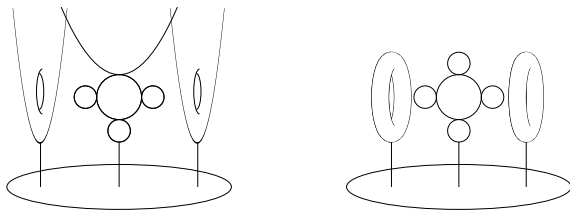


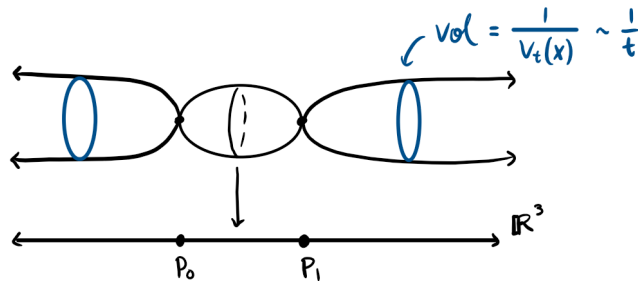
Figure: $\mathcal{X}(\vec{\beta})$ on the left, $\mathcal{M}(\vec{\alpha})$ on the right

Classic Example of Instanton Degeneration

The ALF- A_n instantons have explicit metric descriptions using Gibbons-Hawking potentials on a principal $U(1)$ -bundle over $\mathbb{R}^3$

$$V_t(x) = t + \frac{1}{4\pi \prod_{i=0}^n |x - p_i|}$$

By degenerating $t \rightarrow 0$, the ALF- A_n metrics converge to an ALE- A_n metric.



$$\frac{1}{V_t(x)} = \frac{4\pi \prod |x - p_i|}{4\pi t \prod |x - p_i| + 1}$$

Figure: ALF- A_1 instanton

Bubbling Limits

Gravitational instantons appear as bubbling limits of degenerating K3 surfaces by “zooming in” as a singularity develops.

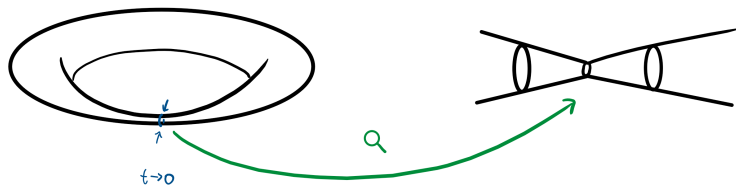


Figure: Example of bubbling limits

Studying degenerations between instantons lends insight towards the boundary faces of the moduli of K3 surfaces.

Theorem 1 is thus one step in a larger story about how the various boundary faces meet each other.

Other Instanton Degenerations

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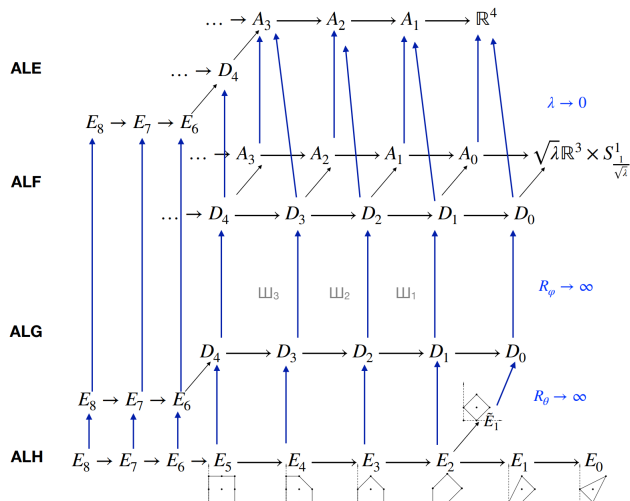


Figure: Conjectured Instanton Degenerations (by Sergey Cherkis)

Theorem Statement

Theorem

As $R \rightarrow 0$, the family of metrics g_R converges pointwise to the metric $2\pi g_{\text{HP}}$ on the restriction to the image of $\mathcal{T}_R : \mathcal{X}(\vec{\beta}) \hookrightarrow \mathcal{M}_R(\vec{\alpha}(R))$. In other words, $\lim_{R \rightarrow 0} \mathcal{T}_R^*(g_R) = 2\pi g_{\text{HP}}$.

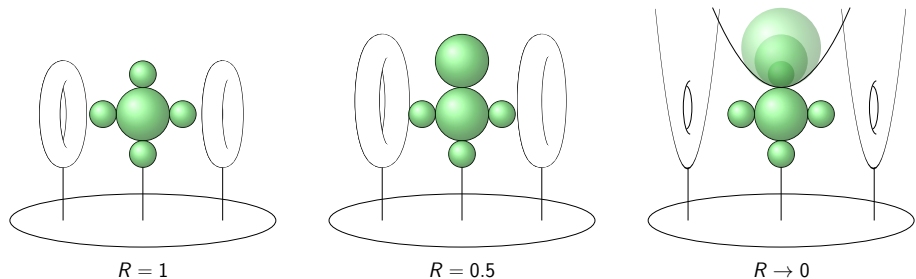


Figure: Degenerate Limit as $R \rightarrow 0$, ($n = 4$ case)

Proof Strategy

To calculate the pullback $\mathcal{T}_R^*(g_R)$, we evaluate g_R on the pushforward of tangent vectors. Given a 1-parameter family $(\mathbf{x}(t), \mathbf{y}(t)) \in \mathcal{X}(\vec{\beta})$, we use $\mathcal{T}_R$ and the NAHC to produce

$$(\mathbf{x}(t), \mathbf{y}(t)) \xrightarrow{\mathcal{T}_R} (\bar{\partial}, \varphi(t), \mathcal{F}(t), h_R(t)) \xrightarrow{\text{gauge change}} (A_R(t), \Phi_R(t), \mathcal{F}(0), h_R(0))$$

which solves Hitchin's equations for all t .

Then we isolate the first variations and calculate

$$\|(\dot{A}_R, \dot{\Phi}_R)\|_{g_R}^2 = \int_C R^{-1} |\dot{A}_R|_{h_R}^2 + R |\dot{\Phi}_R|_{h_R}^2 \xrightarrow[R \rightarrow 0]{(\text{goal})} 2\pi \sum_{i=1}^n |\dot{x}_i|^2 + |\dot{y}_i|^2.$$

Proof Strategy

The proof requires information about the family of harmonic metrics h_R . Solving Hitchin's equation explicitly is infeasible, but the limiting behavior can be understood.

The main challenge was reconciling the effects of degenerating the parameter R in Hitchin's equation with those of taking the limit $\vec{\alpha}(R') \rightarrow (\frac{1}{2}, \dots, \frac{1}{2})$. It is precisely the relative rates at which R and R' approach 0 that encodes the data from $\vec{\beta}$.

We capture data from the relative rates by constructing local model metrics, and we glue them to produce a global metric. Then we perturb to an actual solution using an implicit function theorem argument.

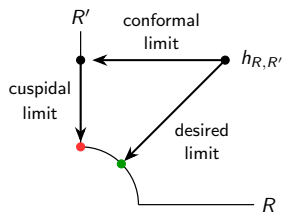


Figure: Micro-Scale Analysis as $R \rightarrow 0$ and $R' \rightarrow 0$

Local Models

Fix $i \in \{1, \dots, n\}$ and recall

$$0 = \mu_{\mathbb{C}^\times, i}(\mathbf{x}, \mathbf{y}) = y_i x_i, \quad \beta_i = \mu_{U(1), i}(\mathbf{x}, \mathbf{y}) = \frac{1}{2}(|x_i|^2 - |y_i|^2)$$

Locally, we have the parabolic structure

$$\begin{array}{ccccc} 0 & \subset & \langle x_i \rangle & \subset & E_{p_i} \\ 1 & > & \frac{1}{2} + R\beta_i & > & \frac{1}{2} - R\beta_i \end{array}$$

and Higgs field

$$\varphi_{\text{loc}} = \frac{x_i y_i}{z - p_i} dz.$$

In coordinates with $x_i \in \langle e_1 \rangle$, $x_i y_i = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$. If we apply the ansatz that

$h_R = \sqrt{h_{\det}} \text{diag}(\lambda_R, \lambda_R^{-1})$, Hitchin's equation reduces to the ODE

$$\bar{\partial} \partial \log \lambda_R = R^2 \frac{|x_i y_i|^2}{r^2} \lambda_R^2 d\bar{z} \wedge dz$$

where $r = |z - p_i|$.

Local Models

$$\bar{\partial} \partial \log \lambda_R = R^2 \frac{|x_i y_i|^2}{r^2} \lambda_R^2 d\bar{z} \wedge dz$$

Generalizing results from [KW18], we obtain the family of local solutions

$$\lambda_R = \frac{2c_i \beta_i r^{2R\beta_i}}{1 - |x_i y_i|^2 c_i^2 r^{4R\beta_i}}, \quad c_i > 0.$$

By setting $c_i = |x_i|^{-2}$ for $i = 1, \dots, n$, we can glue our local models with a globally-defined flat metric to produce an approximation $h_{\text{app}, R}$. The gluing uses the equation

$$0 = \mu_{\text{SU}(2)}(\mathbf{x}, \mathbf{y}) = \sum (x_i x_i^\dagger - y_i^\dagger y_i)_0.$$

The failure for $h_{\text{app}, R}$ to solve Hitchin's equation $R^{-1} F_{\nabla(\bar{\partial}, h_{\text{app}, R})}^\perp + R[\varphi^\dagger_{h_{\text{app}, R}}, \varphi] = 0$ decays as $R \rightarrow 0$. To prove Theorem 1, we must perturb $h_{\text{app}, R}$ to an actual solution for small $R > 0$.

Analytic Implicit Function Theorem

The *b-weighted Sobolev space* $L_\delta^{k,2}(U)$ is the functions which can be written as $r^\delta u$ where $u \in L^{k,2}(U)$ in a neighborhood of the punctures. Here, we use the *b*-vector fields $r\partial_r$ and ∂_θ to define the Sobolev space near marked points $p_i \in D$.

A metric h adapted to the parabolic structure is asymptotically of the form

$$h = \begin{pmatrix} O(r^{2-2\alpha_i}) & O(r^{2\alpha_i}) \\ O(r^{2\alpha_i}) & O(r^{2\alpha_i}) \end{pmatrix} = \begin{pmatrix} O(r^{1+2R\beta_i}) & O(r^{1-2R\beta_i}) \\ O(r^{1-2R\beta_i}) & O(r^{1-2R\beta_i}) \end{pmatrix}, \quad r = |z - p_i|.$$

In general, the error in Hitchin's equation belongs to the weighted Sobolev space $L_{-1-\varepsilon_R}^{0,2}(\Omega_C^{1,1} \otimes i\text{End}_0(E, h))$ for some $0 < \varepsilon_R < 4R\beta_i$.

Because of the careful gluing procedure, our $h_{\text{app},R}$ produces an error in the subspace $L_{-\varepsilon}^{0,2}(\Omega_C^{1,1} \otimes i\text{End}_0(E, h_{\text{app},R}))$ for a fixed $\varepsilon \in (0, \frac{1}{2})$.

Analytic Implicit Function Theorem

A section $\gamma \in L_{2-\varepsilon}^{2,2}(i\text{End}_0(E, h_0))$ is used to map $h_{\text{app},R} \mapsto e^{\gamma^\dagger} h_{\text{app},R} e^\gamma$, which we follow by a gauge transformation

$$(\bar{\partial}, \varphi, e^{\gamma^\dagger} h_{\text{app},R} e^\gamma) \mapsto (A_{R,\gamma}, \Phi_{R,\gamma}, h_0).$$

The map

$$\mathcal{N}_R : L_{2-\varepsilon}^{2,2}(i\text{End}_0(E, h_0)) \rightarrow L_{-\varepsilon}^{0,2}(\Omega_C^{1,1} \otimes i\text{End}_0(E, h_0))$$

given by

$$\mathcal{N}_R(\gamma) = F_{(A_{R,\gamma}, h_0)}^\perp + R^2 \left[\Phi_{R,\gamma}^{\dagger h_0}, \Phi_{R,\gamma} \right]$$

satisfies the following properties:

1. $\mathcal{N}_R(\gamma) = 0$ if and only if $(\bar{\partial}, \varphi, e^{\gamma^\dagger} h_{\text{app},R} e^\gamma)$ solves Hitchin's equation;
2. $\lim_{R \rightarrow 0} \mathcal{N}_R(0) = 0$;
3. The linearization $d\mathcal{N}_R|_{\gamma=0}$ is injective when $R = 0$ and has Fredholm index 0.

The analytic implicit function theorem produces a family of corrections

$\gamma_R \in L_{2-\varepsilon}^{2,2}(i\text{End}_0(E, h_0))$ for $R \in (0, R_{\max}]$ such that $\lim_{R \rightarrow 0} \gamma_R = 0$.

Proof of Main Theorem

Given a 1-parameter family $(\mathbf{x}(t), \mathbf{y}(t)) \in \mathcal{X}(\vec{\beta})$, we use $\mathcal{T}_R$ and the NAHC to produce

$$(\mathbf{x}(t), \mathbf{y}(t)) \xrightarrow{\mathcal{T}_R} (\bar{\partial}, \varphi(t), \mathcal{F}(t), h_R(t)) \xrightarrow{\text{gauge change}} (A_R(t), \Phi_R(t), \mathcal{F}(0), h_R(0))$$

which solves Hitchin's equations for all t .

Then we isolate the first variations and calculate

$$\|(\dot{A}_R, \dot{\Phi}_R)\|_{\mathcal{G}_R}^2 = \int_C R^{-1} |\dot{A}_R|_{h_R}^2 + R |\dot{\Phi}_R|_{h_R}^2 \xrightarrow[R \rightarrow 0]{(\text{goal})} 2\pi \sum_{i=1}^n |\dot{x}_i|^2 + |\dot{y}_i|^2.$$

The calculation is long and uses each of the following:

- The linearization of Hitchin's equations, which gives relations between the components of $\dot{A}_R$ and $\dot{\Phi}_R$;
- An application Stokes' theorem to isolate the data near the punctures $p_i \in D$;
- The high regularity of our correction γ_R to $h_{\text{app}, R}$ for $R \in (0, R_{\max}]$;
- The local models used in the construction of $h_{\text{app}, R}$.

Theorem Statement

Theorem

As $R \rightarrow 0$, the family of metrics g_R converges pointwise to the metric $2\pi g_{\text{HP}}$ on the restriction to the image of $\mathcal{T}_R : \mathcal{X}(\vec{\beta}) \hookrightarrow \mathcal{M}_R(\vec{\alpha}(R))$. In other words, $\lim_{R \rightarrow 0} \mathcal{T}_R^*(g_R) = 2\pi g_{\text{HP}}$.

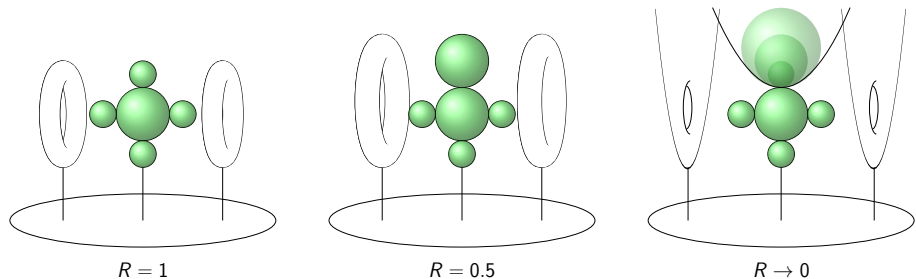


Figure: Degenerate Limit as $R \rightarrow 0$, ($n = 4$ case)

Thank you!

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